

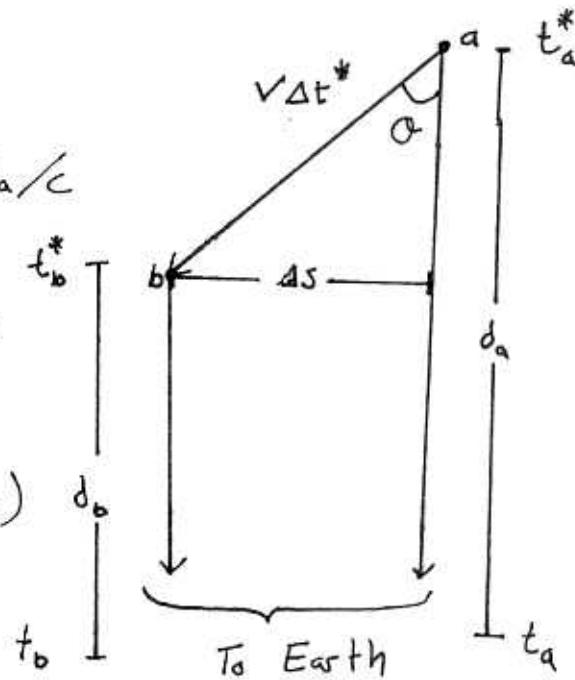
Physics 110B  
Homework #4

#1 (Griffiths 12.6)

- Light signal leaves a at time  $t_a^*$ ; arrives at Earth at time  $t_a = t_a^* + d_a/c$
- Light signal leaves b at time  $t_b^*$ ; arrives at Earth at time  $t_b = t_b^* + d_b/c$

Thus,

$$\begin{aligned}\Delta t &= t_b - t_a = (t_b^* + d_b/c) - (t_a^* + d_a/c) \\ &= t_b^* - t_a^* + (d_b - d_a)/c \\ &= \Delta t^* + (-v\Delta t \cos\theta)/c \\ &= \Delta t^*(1 - v/c \cos\theta)\end{aligned}$$



$d_a$  is the distance from a to Earth  
and  $d_b$  is the distance from b to Earth

$$\Delta s = v\Delta t^* \sin\theta = \frac{v \sin\theta \Delta t}{(1 - v/c \cos\theta)}$$

Therefore, the apparent velocity is,

$$V_{\text{apparent}} = \boxed{\frac{\Delta s}{\Delta t} = \frac{v \sin\theta}{(1 - v/c \cos\theta)}} \quad \text{is the apparent velocity}$$

The maximum value we get by taking the derivative and setting it equal to zero:

$$\frac{dV_{\text{apparent}}}{d\theta} = \frac{v((1 - v/c \cos\theta) \cos\theta - \sin\theta \frac{v}{c} \sin\theta)}{(1 - v/c \cos\theta)^2} = 0$$

$$\Rightarrow (1 - v/c \cos\theta) \cos\theta = v/c \sin^2\theta$$

$$\cos\theta = \frac{v}{c} (\cos^2\alpha + \sin^2\alpha)$$

$$\cos\theta = v/c$$

$$\boxed{\theta_{\max} = \cos^{-1}(v/c)}$$

At the max angle,  $v_{\text{apparent}} = \frac{v\sqrt{1-v^2/c^2}}{1-v^2/c^2} = \frac{v}{\sqrt{1-v^2/c^2}}$

as  $v \rightarrow c$ ,  $v_{\text{apparent}} \rightarrow \infty$  even though  $v < c$ , therefore we see that the apparent velocity can be larger than the speed of light.

#2 (Griffiths 12.18)

a)  $\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$   
 III  
 M<sub>Galilean</sub>

b)  $\Lambda = \boxed{\begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$

c)  $\Lambda = \boxed{\begin{pmatrix} \gamma & 0 & -\gamma\beta' & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta' & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$

$$= \boxed{\begin{pmatrix} \gamma\gamma' & -\gamma\gamma'\beta & -\gamma\beta' & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\gamma'\gamma\beta' & \gamma\gamma'\beta\beta' & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

• Yes, the order of applying the matrices does matter. If switched the orders primes and unprimed would be switched giving us a different matrix

#3 (Griffiths 12.19)

a)  $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$  (eq 12.24)

$$\tanh\alpha \equiv v/c \quad (\text{eq } 12.34)$$

$$\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha}, \quad \cosh^2\alpha - \sinh^2\alpha = 1$$

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \tanh^2\alpha}} = \frac{\cosh\alpha}{\sqrt{\cosh^2\alpha - \sinh^2\alpha}} = \cosh\alpha$$

$$\gamma\beta = \cosh\alpha \tanh\alpha = \sinh\alpha$$

Therefore,

$$\boxed{\Lambda = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

compare with  
Rotation Matrix

$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

b) Einstein's velocity addition rule

$$u' = \frac{u - v}{1 - uv/c^2} \quad (\text{eq } 12.20) \Rightarrow \frac{u'}{c} = \frac{u/c - v/c}{1 - uv/c^2}$$

$$\Rightarrow \tanh\varphi' = \frac{\tanh\varphi - \tanh\alpha}{1 - \tanh\varphi \tanh\alpha}, \quad \text{where } \tanh\varphi \equiv u/c, \quad \tanh\alpha \equiv v/c$$

$\tanh\varphi' = \tanh(\varphi - \alpha)$  using a hyperbolic trig. identity

or R:  $\boxed{\varphi' = \varphi - \alpha}$

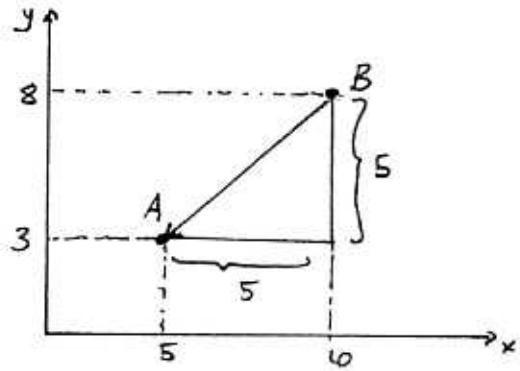
#4 (Griffiths 12.20) (Note: using Prof. Stravink notation w/  $(1, -1, -1, -1)$  signature)

a) i)  $I = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = (5-15)^2 - (10-5)^2 - (8-3)^2 - (0-0)^2$   
 $= 100 - 25 - 25 = \boxed{50}$  timeline

ii) If the events occur simultaneously  $\Delta t' = 0$ , "I" in this case would be negative, which it isn't. Thus, the events cannot occur simultaneously.

iii) S' travels in the direction from B toward A, making the trip in time  $\Delta t = 10/c$ , if the moving frame does this then the two events will occur at the same place though at different times,

$$\bar{v} = -\frac{\Delta x - \Delta y}{\Delta t} = -\frac{c}{2}\hat{x} - \frac{c}{2}\hat{y}$$



We note  $v^2/c^2 = \frac{1}{2} \Rightarrow v = \frac{1}{\sqrt{2}}c$  which has  $v < c$  as has to be

b) i)  $I = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = (3-1)^2 - (5-2)^2 + 0 + 0 = \boxed{-5}$  spacelike

ii) Yes, can occur simultaneously. We want  $\Delta t' = 0$ , therefore using the Lorentz transformation,

$$c \Delta t' = \gamma(c \Delta t - \beta \Delta x) = 0 \Rightarrow c \Delta t = \beta \Delta x$$

Or,  $v/c = c \Delta t / \Delta x = \cancel{(3-1)} / (5-2) = \frac{2}{3}$  So,  $V = \frac{2}{3}c$  in the x direction

iii) No, the events cannot occur at the same location since this would require that  $\Delta x' = \Delta y' = \Delta z' = 0$  so that "I" would be positive which it is not.

#5 (Griffiths 12.32)

$$\stackrel{\longleftarrow}{\textcircled{1}} \quad + \quad \stackrel{\textcircled{2}}{m} \longrightarrow \quad \textcircled{3}$$

$$E_1 = 2mc^2$$

$$E_3, M_3, P_{3x}$$

$$P_1 = (2mc, P_{x1}, 0, 0) \quad P_3 = (E_3, P_{3x}, 0, 0)$$

$$P_2 = (mc, 0, 0, 0)$$

- $P_1 + P_2 = P_3$

$$(P_1 + P_2)^2 = P_3^2$$

$$P_1 \cdot P_1 + 2P_1 \cdot P_2 + P_2 \cdot P_2 = P_3 \cdot P_3$$

$$m^2 c^2 + 2(2m^2 c^2) + m^2 c^2 = m_3^2 c^2$$

$$\boxed{m_3 = \sqrt{6} m}$$

- $P_1 \cdot P_1 = 4m^2 c^2 - P_{x1}^2 = m^2 c^2 \Rightarrow P_{x1} = \sqrt{3} mc$

$$\bar{P}_{T\text{initial}} = P_{x1} + P_{x2} = \sqrt{3} mc$$

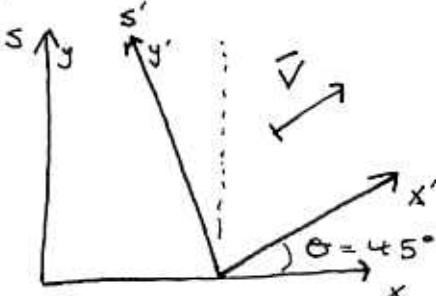
$$E_{T\text{initial}} = E_1 + E_2 = 3mc^2$$

$$\bar{V}_{CM} = \frac{\bar{P}_{Ti} c^2}{E_{Ti}} = \frac{(\sqrt{3} mc)c^2}{3mc^2} = \boxed{\frac{1}{\sqrt{3}} c = V_{CM}}$$

6(a)

$$\mathcal{L} = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)



$$\bar{v} = \frac{\beta c}{\sqrt{2}} (\hat{x} + \hat{y}) = \beta c (\cos 45^\circ, \sin 45^\circ)$$

$$\mathcal{L}_{\text{total}} = R^{-1} \mathcal{L} R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta\cos\alpha & -\gamma\beta\sin\alpha \\ -\gamma\beta & \gamma\cos\alpha & \gamma\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta\cos\alpha & -\gamma\beta\sin\alpha \\ -\gamma\beta\cos\alpha & \gamma\cos^2\alpha + \sin^2\alpha & \gamma\cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ -\gamma\beta\sin\alpha & \gamma\cos\alpha\sin\alpha & \gamma\sin^2\alpha + \cos^2\alpha \end{pmatrix}$$

$$\mathcal{L}_{\text{Total}} = \begin{pmatrix} \gamma & -\gamma\beta/\sqrt{2} & -\gamma\beta/\sqrt{2} \\ -\gamma\beta/\sqrt{2} & (1+\gamma)/2 & (\gamma-1)/2 \\ -\gamma\beta/\sqrt{2} & (\gamma-1)/2 & (1+\gamma)/2 \end{pmatrix}$$

• as  $\beta \rightarrow 0$      $\gamma \rightarrow 1$

$$\mathcal{L}_{\text{total}} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

• Also, the result is symmetric under the interchange of  $x$  and  $y$ .

#7.

$$\overset{\rightarrow}{P}_1 + \overset{\rightarrow}{P}_2 \longrightarrow \overset{\rightarrow}{P} + \overset{\rightarrow}{P} + \overset{\rightarrow}{P} + \overset{\rightarrow}{P}$$

$E_b$

final state  
moving together uniformly

$$P_1 = (E_b/c, P_{1x}, 0, 0) \quad P'_{FT} = (4m_p c, 0, 0, 0)$$

$$P_2 = (m_p c, 0, 0, 0)$$

$$P_1 + P_2 = P_{\text{final total}}$$

$$(P_1 + P_2) \cdot (P_1 + P_2) = P_{FT} \cdot P_{FT} = P'_F \cdot P'_F = 16 m_p^2 c^2$$

$$P_1 \cdot P_1 + 2 P_1 \cdot P_2 + P_2 \cdot P_2 = 16 m_p^2 c^2$$

$$m_p^2 c^2 + 2 m_p E_b + m_p^2 c^2 = 16 m_p^2 c^2$$

$$E_b = 7 m_p c^2 = 7 \times (9.4 \times 10^9 \text{ eV})$$

$$E_{\text{beam}} = 6.58 \times 10^9 \text{ eV}$$

#8.

$$(a) \quad V = g t = (9.8) \cdot (10) \cdot (31.6 \times 10^{-6}) = \boxed{3.10 \times 10^9 \text{ m/s}}$$

$$(b) \quad g d\tau = c d\beta = c d(\tanh \gamma) = (c / \cosh^2(\eta \approx 0)) d\gamma \approx c d\eta$$

$$c d\eta = g d\tau$$

$$\Rightarrow \int c d\eta = \int g d\tau \Rightarrow \boxed{c\eta = g\tau}$$

$$(C) \quad \beta = \tanh \gamma \Rightarrow \frac{dx}{dt} = c \tanh \gamma$$

$$\Rightarrow dx = c \tanh \gamma dt$$

$$\text{Now, } dt = \gamma d\tau = \cosh \gamma d\tau$$

$$\Rightarrow dx = c \tanh \gamma \cosh \gamma d\tau = c \sinh \gamma d\tau$$

$$\text{Using } c\gamma = g \tau$$

$$dx = c \sinh \left( \frac{g\tau}{c} \right) d\tau$$

$$\int dx = \int c \sinh \left( \frac{g\tau}{c} \right) d\tau$$

$$\Rightarrow \boxed{x = \frac{c^2}{g} \left( \cosh \left( \frac{g\tau}{c} \right) - 1 \right)}$$

(d) The maximum amount of time the astronaut can travel is  $T_m = 40$  yrs.  
in a quarter of that time she travels  $\tau = 10$  yrs.

Now, figuring the x-distance for this time:

$$x = \frac{c^2}{g} \left( \cosh \left( \frac{g\tau}{c} \right) - 1 \right) = \frac{(2.998 \times 10^8)^2}{9.8} \left( \cosh \left( \frac{9.8 \cdot 10 \cdot 31.6 \times 10^6}{2.998 \times 10^8} \right) - 1 \right)$$

$$= 1.404 \times 10^{20} \text{ m}$$

$$x_{\text{total}} = 2x = 2.8 \times 10^{20} \text{ m} \cdot \frac{1 \text{ lyr}}{31.6 \times 10^6 \text{ s} \cdot 3 \times 10^8 \text{ m/s}}$$

$$\boxed{x_{\text{total max}} = 29,650 \text{ lightyears}}$$

$x_{\text{total max out}} > 7000$  lightyears. Thus, the Engineer was correct.